

## Modeling the Stackelberg strategy in a linear model (linear city) Hotteling

<sup>1</sup>Musayeva Shoira Azimovna,

<sup>2</sup>Usmonova Dilfuza Ilkhomovna

<sup>3</sup>Usmanov Farzod Shokhrukhovich

<sup>1</sup> Professor of Samarkand Institute of Economic and Service, Samarkand, Uzbekistan

<sup>2</sup>PhD doctor , Samarkand Institute of economic and services, Samarkand, Uzbekistan

<sup>3</sup>Student, Samarkand Institute of economic and services, Samarkand, Uzbekistan

<sup>1</sup>E-mail: [musaeva\\_shoira@mail.ru](mailto:musaeva_shoira@mail.ru)

\*\*\*

**Abstract.** *The article examines the model of a linear city with exogenous Stackelberg competition between two firms. In this model, at low transport costs, firms in equilibrium are located at one point in the center of the market, while the profit of the leader firm is twice the profit of the follower firm, the price is minimal, and the quantity of products supplied is maximal at the point where the firms are located. At higher transport costs, firms differentiate, and the market, as it were, splits into two "submarkets": the leader firm sells the bulk of production near its location, and the follower company sells the bulk of its products, while the profit of the leader firm exceeds the profit of the follower firm by less than twice, the price is always minimal at the point of location of the leading firm. With an increase in transport costs, the quantities of products supplied by firms decrease, and the price rises.*

**Key words:** *spatial competition, Stackelberg oligopoly, Hotelling's linear city model.*

### Introduction.

Definition of the linear model (linear city) Hotteling. "Suppose that along a straight beach of

Let's make the following designation according to Figure 2.

Let  $a + x + y + b = L$

length  $L$ , at a distance  $a$  and  $b$  from its left and right ends, there are 2 trays -  $A$  and  $B$ , from which ice cream cones are sold. Buyers are placed at a distance of a unit of length from each other, and each buys one horn within a given period of time. The cost of producing ice cream is zero, and the cost of its "transportation" by the buyer from the tray to his place under the beach umbrella is equal to  $c$  per one unit of the way (after all, ice cream melts on the way!) Let  $P_A$  be the price of one cone on the tray  $A$ ,  $P_B$  - the price of one cone on Tray  $B$ .

Then the buyer at point  $E$  does not care which of the two trays to buy ice cream from, if the following condition is met:

$$P_A + cx = P_B + cy \tag{1}$$

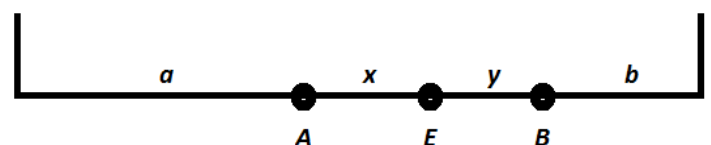


Fig 1. Model of the linear city "Hotteling"

Where L is the length of the beach.

Find the coordinates of the point E, determining the distances x and y:

$$X = \frac{P_b - P_a + cy}{c} = \frac{P_b - P_a}{c} + L - a - b - x \quad (2)$$

or

$$X = \frac{1}{2} * \left( L - a - b + \frac{P_b - P_a}{c} \right)$$

$$Y = \frac{1}{2} * \left( L - a - b + \frac{P_a - P_b}{c} \right)$$

Let's write expressions for the profit of each of the trays (i.e., each of the firms):

$$\Pi_a = P_a(a + x) = \frac{1}{2} * \left( (L - a - b)P_a + \frac{P_a P_b - P_a^2}{c} \right)$$

$$\Pi_b = P_b(b + y) = \frac{1}{2} * \left( (L - a + b)P_b + \frac{P_a P_b - P_b^2}{c} \right)$$

Each of the firms will choose a price that maximizes its profits, based on the necessary condition for maximizing profits:

$$\frac{\partial \Pi_a}{\partial P_a} = \frac{1}{2} * \left( (L - a - b) + \frac{P_b - 2 * P_a}{c} \right) = 0$$

$$\frac{\partial \Pi_b}{\partial P_b} = \frac{1}{2} * \left( (L - a - b) + \frac{P_a - 2 * P_b}{c} \right) = 0$$

By solving the equations together, we find that:

$$P_a = c \left( L + \frac{a - b}{3} \right)$$

$$P_b = c \left( L - \frac{a - b}{3} \right)$$

If the stalls (firms) are located at different distances from the ends of the beach, then these prices will differ from each other, and the differences in prices, due to the identity and zero marginal production costs, are explained solely by the difference in location: the firm that is better located may require a

higher price for the product, while retaining a larger market share, that is, not yielding any significant market share to another company.

Note that this model does not entail the strategic behavior of the two firms. Below we will consider a deeper analysis of the linear model in terms of the Stackelberg model.

Main part.

### **Stackelberg model of a linear city with exogenous competition.**

Definition: "Let the following assumptions be fulfilled"

1. The market is a unit segment  $[0, 1]$  with a uniform distribution of consumers.
2. Firm-leader (firm 1) is located at point  $x_1$ , and the follower firm (firm 2) – at point  $x_2$ , and firm 1 is located "To the left" of firm 2, that is:  $x_1 \leq x_2$
3. *The function of transport costs is linear in  $x$ , that is, transport costs for the delivery of a unit of goods from the location of firm  $k$  to point  $x$  is determined as follows*

$$t|x - x_k|$$

Where  $t$  – transportation costs for the transportation of a unit of production

4. *There is no cooperative behavior of firms*
5. *In the first round, firms choose a location, in the second round - the number of products offered.*
6. *Firm 1 is the leader in both rounds and anticipates the reaction of firm 2, which is the follower.*
7. *Production costs of both firms are zero.*
8. *Firms themselves pay transport costs for the delivery of goods to the consumer.*
9. *Each point of the market is a separate submarket, where a different price can be established from the price in other submarkets, that is, there can be price discrimination based on location.*

Let the demand function at point  $x$  be defined as

$$P = 1 - Q$$

Where  $P$  is the price,  $Q$  is the quantity of goods. Let

$$Q = q_1 + q_2$$

Where  $q_1$  and  $q_2$  are the quantities of goods supplied by firms 1 and 2, respectively.

#### **Let's start by looking at the second round:**

Let us define the profit of the **follower firm** at point  $x$ :

$$\Pi_2(x_1, x_2, q_1, q_2) = q_2(1 - q_1 - q_2) - q_2 t|x - x_2| = -q_2 t|x - x_2| - q_2^2 + (-q_1 + 1)q_2$$

Let's define the profit of the **company - the leader** at the point  $x$ :

$$\Pi_1(x_1, x_2, q_1, q_2) = q_1(1 - q_1 - q_2) - q_1 t |x - x_1| = -q_1 t |x - x_1| - q_1^2 + (-q_1 + 1)q_1$$

Let us find the reaction function by the number of the follower firm in the second round. For this, the derivative of the function  $\Pi_2(x_1, x_2, q_1, q_2)$  for  $q_2$  equate to zero:

$$\partial \pi_2(x_1, x_2, q_1, q_2) / \partial q_2 = -t |x_2 - x| - 2q_2 - q_1 + 1 = 0.$$

Solving this equation, we obtain the reaction function for the number of firms 2  $q_2(q_1)$ :

$$q_2(q_1) = -\frac{t |x_2 - x| + q_1 - 1}{2}.$$

Substituting the response function of the follower firm (the above equation) into the profit function of the leader's firm:

$$\begin{aligned} \pi_1(x_1, x_2, q_1, q_2) &= \\ &= (-q_1) t |x_1 - x| - q_1 \frac{-(t |x_2 - x| + q_1 - 1)}{2} - q_1^2 + q_1 = \\ &= \frac{q_1 t |x_2 - x| - 2q_1 t |x_1 - x| - q_1^2 + q_1}{2}. \end{aligned}$$

To find the equilibrium quantity of goods offered by firms at the point  $x$ , we calculate the derivative of the above function with respect to  $q_1$ :

$$\partial \pi_1(x_1, x_2, q_1, q_2) / \partial q_1 = \frac{t |x_2 - x| - 2t |x_1 - x| - 2q_1 + 1}{2}.$$

Equating to zero and solving the resulting equation, we obtain the equilibrium quantity of goods supplied by the leader firm at the point  $x$ ;

$$q_1^*(x, x_1, x_2) = \frac{t |x_2 - x| - 2t |x_1 - x| + 1}{2}.$$

Substituting the above equation into the function  $q_2(q_1)$  we find the equilibrium quantity of goods supplied at point  $x$  by a firm follower of  $q_2^*(x, x_1, x_2)$ :

$$q_2^*(x, x_1, x_2) = \frac{-\frac{t|x_2-x|-2t|x_1-x|+1}{2} - t|x_2-x| + 1}{2} =$$

$$= -\frac{3t|x_2-x| - 2t|x_1-x| - 1}{4}.$$

Using  $P = 1 - Q$  and  $Q = q_1 + q_2$ , we find the equilibrium price  $p^*(x, x_1, x_2)$  at the point  $x$ :

$$p^*(x, x_1, x_2) =$$

$$= -\frac{t|x_2-x| - 2t|x_1-x| + 1}{2} - \frac{-3t|x_2-x| + 2t|x_1-x| + 1}{4} + 1 =$$

$$= \frac{t|x_2-x| + 2t|x_1-x| + 1}{4}. \quad (2.13)$$

Substituting in  $\Pi_2(x_1, x_2, q_1, q_2)$  and  $\Pi_1(x_1, x_2, q_1, q_2)$  the equilibrium values  $q_1^*(x, x_1, x_2)$  и  $q_2^*(x, x_1, x_2)$ , we find the equilibrium profit of firm 1  $\pi_1^*$  and and equilibrium profit  $\pi_2^*$  of firm 2 at point  $x$ .

$$\pi_2^*(x, x_1, x_2) =$$

$$= \frac{1}{16} \left( 9t^2|x_2-x|^2 + (-12t^2|x_1-x| - 6t)|x_2-x| + \right.$$

$$\left. + 4t^2|x_1-x|^2 + 4t|x_1-x| + 1 \right),$$

$$\pi_1^*(x, x_1, x_2) =$$

$$= \frac{1}{8} \left( t^2|x_2-x|^2 + (-4t^2|x_1-x| + 2t)|x_2-x| + \right.$$

$$\left. + 4t^2|x_1-x|^2 - 4t|x_1-x| + 1 \right).$$

**Round one:**

Let us find the integral function of the profit of the firm - the follower of  $\Pi_2(x_1, x_2)$ . By the integral function we mean a function that describes the market as a whole, and not a single point in the market.

$$\begin{aligned} \Pi_2(x_1, x_2) &= \int_0^1 \pi_2^*(x, x_1, x_2) dx = \\ &= \frac{1}{16} \int_0^1 \left( (-12t^2 |x_1 - x| - 6t) |x_2 - x| + 9t^2 (x_2 - x)^2 + \right. \\ &\quad \left. + 4t |x_1 - x| + 4t^2 (x_1 - x)^2 + 1 \right) dx. \end{aligned}$$

Given the constraints on the relative position of firms, the expression for the integral function shown above can be calculated as follows:

$$\begin{aligned} \Pi_2(x_1, x_2) &= \int_0^1 \pi_2^*(x, x_1, x_2) dx = \\ &= \int_0^{x_1} \pi_2^*(x, x_1, x_2) dx + \int_{x_1}^{x_2} \pi_2^*(x, x_1, x_2) dx + \\ &\quad + \int_{x_2}^1 \pi_2^*(x, x_1, x_2) dx = \\ &= -\frac{1}{48} \left( 12t^2 x_2^3 + (-36t^2 x_1 - 27t^2 + 18t) x_2^2 + \right. \\ &\quad + (36t^2 x_1^2 + 36t^2 x_1 + 9t^2 - 18t) x_2 + \\ &\quad - 12t^2 x_1^3 + (-12t^2 - 12t) x_1^2 + (12t - 6t^2) x_1 - t^2 + \\ &\quad \left. + 3t - 3 \right). \end{aligned}$$

Let us construct a reaction function based on the location of the follower firm  $x_2(x_1)$ . Find the value  $x_2$ , that maximizes the follower firm's response function. Let us find the derivative  $\Pi_2(x_1, x_2)$ :

$$\begin{aligned} \partial \Pi_2(x_1, x_2) / \partial x_2 &= \\ &= -\frac{1}{16} \left( 12t^2 x_2^2 + (-24t^2 x_1 - 18t^2 + 12t) x_2 + \right. \\ &\quad \left. + 12t^2 x_1^2 + 12t^2 x_1 + 3t^2 - 6t \right). \end{aligned}$$

Equating the equation to zero and solving the resulting equations, we get:

$$x_2(x_1) = -\frac{\sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} - 4t x_1 - 3t + 2}{4t}$$

and

$$x_2(x_1) = \frac{\sqrt{(8t^2 - 16t)x_1 + 5t^2 - 4t + 4} + 4tx_1 + 3t - 2}{4t}.$$

Let us find the integral function of the profit of the company - leader  $\Pi_1(x_1, x_2)$ .

$$\begin{aligned} \Pi_1(x_1, x_2) &= \int_0^1 \pi_1^*(x, x_1, x_2) dx = \\ &= \frac{1}{8} \int_0^1 \left( (2t - 4t^2(x - x_1)) |x_2 - x| + t^2(x_2 - x)^2 + \right. \\ &\quad \left. - 4t(x - x_1) + 4t^2(x - x_1)^2 + 1 \right) dx. \end{aligned}$$

Let's write the integral function as a sum:

$$\begin{aligned} \Pi_1(x_1, x_2) &= \int_0^1 \pi_1^*(x, x_1, x_2) dx = \\ &= \int_0^{x_1} \pi_1^*(x, x_1, x_2) dx + \int_{x_1}^{x_2} \pi_1^*(x, x_1, x_2) dx + \\ &\quad + \int_{x_2}^1 \pi_1^*(x, x_1, x_2) dx. \end{aligned}$$

Then:

$$\begin{aligned} \Pi_1(x_1, x_2) &= \\ &= -\frac{1}{24} \left( 4t^2 x_2^3 + (-12t^2 x_1 - 3t^2 - 6t) x_2^2 + \right. \\ &\quad \left. + (12t^2 x_1^2 + 12t^2 x_1 - 3t^2 + 6t) x_2 + \right. \\ &\quad \left. - 4t^2 x_1^3 + (12t - 12t^2) x_1^2 + (6t^2 - 12t) x_1 - t^2 + 3t - 3 \right). \end{aligned}$$

Substituting the response functions of the follower firm by location X2 (X1) into the integral profit function of the leader firm (indicated above).

When setting the First X2 (X1), we get:

$$\begin{aligned} \Pi_1(x_1) &= \\ &= \frac{1}{192t} \left( (24t^3 - 48t^2) x_1^2 + \right. \\ &\quad \left. + \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} \times \right. \\ &\quad \left. \times ((16t^2 - 32t) x_1 + t^2 - 20t + 20) + \right. \\ &\quad \left. + (-84t^3 + 240t^2 - 144t) x_1 + 11t^3 + 6t^2 - 36t + 40 \right). \end{aligned}$$

When setting the Second X2 (X1) we get:

$$\begin{aligned} \Pi_1(x_1) &= \\ &= -\frac{1}{192t} \left( (48t^2 - 24t^3) x_1^2 + \right. \\ &\quad \left. + \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} \times \right. \\ &\quad \left. \times ((16t^2 - 32t) x_1 + t^2 - 20t + 20) + \right. \\ &\quad \left. + (84t^3 - 240t^2 + 144t) x_1 - 11t^3 - 6t^2 + 36t - 40 \right). \end{aligned}$$

Having received the equations of profit, we can then proceed to the analysis of the extrema of each of them:

$$\begin{aligned} \Pi_1(x_1) &= \\ &= \frac{1}{192t} \left( (24t^3 - 48t^2) x_1^2 + \right. \\ &\quad \left. + \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} \times \right. \\ &\quad \left. \times ((16t^2 - 32t) x_1 + t^2 - 20t + 20) + \right. \\ &\quad \left. + (-84t^3 + 240t^2 - 144t) x_1 + 11t^3 + 6t^2 - 36t + 40 \right). \end{aligned}$$

**First Option:** Function

Reaches an extremum at a point

$$\begin{aligned} x_1 &= \\ &= \frac{(7t - 6) \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} - 7t^2 + 12t - 12}{4t \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} + 16t^2 - 32t}. \end{aligned} \tag{A}$$



Let's replace the variable in  $x_1$  with the one shown above:

$$w = \sqrt{(8t^2 - 16t)x_1 + 5t^2 - 4t + 4} \quad w > 0. \tag{B}$$

Then:

$$x_1 = \frac{w^2 - 5t^2 + 4t - 4}{8t^2 - 16t}. \tag{C}$$

Substituting (C) in (A) and solve this equation for  $w$

We get:

$$w = -\frac{\sqrt{41t^2 - 140t + 132} + 7t - 10}{2}, \tag{D}$$

$$w = \frac{\sqrt{41t^2 - 140t + 132} - 7t + 10}{2} \tag{E}$$

And

$$w = 3t - 2. \tag{F}$$

In (C) we put (D), (E) and (F). We get:

$$x_1 = \frac{(7t - 10) \sqrt{41t^2 - 140t + 132} + 35t^2 - 132t + 108}{16t^2 - 32t},$$

$$x_1 = -\frac{(7t - 10) \sqrt{41t^2 - 140t + 132} - 35t^2 + 132t - 108}{16t^2 - 32t}$$

and

$$x_1 = \frac{1}{2}.$$

Let's find the X2 values by the above three roots X1. To do this, we substitute these roots in the

$$x_2(x_1) = -\frac{\sqrt{(8t^2 - 16t)x_1 + 5t^2 - 4t + 4} - 4tx_1 - 3t + 2}{4t}$$

equation:

Below we get the following roots:

$$\begin{aligned} x_2 &= \\ &= -\frac{1}{2^{\frac{9}{2}}t^2 - 2^{\frac{11}{2}}t} \times \\ &\times \left( (4t - 8) \sqrt{(7t - 10) \sqrt{41t^2 - 140t + 132} + 45t^2 - 140t + 116} + \right. \\ &\quad \left. + \left( 5 \cdot 2^{\frac{3}{2}} - 7\sqrt{2}t \right) \sqrt{41t^2 - 140t + 132} - 47\sqrt{2}t^2 + \right. \\ &\quad \left. + 41 \cdot 2^{\frac{5}{2}}t - 31 \cdot 2^{\frac{5}{2}} \right), \quad (2.34) \end{aligned}$$



$$\begin{aligned} x_2 &= \\ &= -\frac{1}{2^{\frac{9}{2}}t^2 - 2^{\frac{11}{2}}t} \times \\ &\times \left( (4t - 8) \sqrt{(10 - 7t) \sqrt{41t^2 - 140t + 132} + 45t^2 - 140t + 116} + \right. \\ &\quad \left. + \left( 7\sqrt{2}t - 5 \cdot 2^{\frac{3}{2}} \right) \sqrt{41t^2 - 140t + 132} - 47\sqrt{2}t^2 + \right. \\ &\quad \left. + 41 \cdot 2^{\frac{5}{2}}t - 31 \cdot 2^{\frac{5}{2}} \right) \quad (2.35) \end{aligned}$$

$$\text{H} \quad x_2 = \frac{1}{2}. \quad (2.36)$$

**Second option:**

Function:

$$\begin{aligned} \Pi_1(x_1) = & \\ & = -\frac{1}{192t} \left( (48t^2 - 24t^3) x_1^2 + \right. \\ & \quad \left. + \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} \times \right. \\ & \quad \left. \times ((16t^2 - 32t) x_1 + t^2 - 20t + 20) + \right. \\ & \quad \left. + (84t^3 - 240t^2 + 144t) x_1 - 11t^3 - 6t^2 + 36t - 40 \right). \end{aligned}$$

$$\begin{aligned} x_1 = & \\ & = \frac{(7t - 6) \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} + 7t^2 - 12t + 12}{4t \sqrt{(8t^2 - 16t) x_1 + 5t^2 - 4t + 4} - 16t^2 + 32t}. \end{aligned}$$

Reaches an extremum at a point

By a similar change of variable we obtain

$$x_1 = -\frac{(7t - 10) \sqrt{41t^2 - 140t + 132} - 35t^2 + 132t - 108}{16t^2 - 32t},$$

$$x_1 = \frac{(7t - 10) \sqrt{41t^2 - 140t + 132} + 35t^2 - 132t + 108}{16t^2 - 32t}$$

И

$$x_1 = \frac{1}{2}.$$

CH

Let's find the X2 values by the above three roots X1.

$$x_2 = -\frac{1}{16t^2 - 32t} \times \left( (4 - 2t) \left| \sqrt{41t^2 - 140t + 132} - 7t + 10 \right| + (7t - 10) \sqrt{41t^2 - 140t + 132} - 47t^2 + 164t - 124 \right),$$

$$x_2 = \frac{1}{16t^2 - 32t} \times \left( (2t - 4) \left| \sqrt{41t^2 - 140t + 132} + 7t - 10 \right| + (7t - 10) \sqrt{41t^2 - 140t + 132} + 47t^2 - 164t + 124 \right)$$

И

$$x_2 = \frac{|3t - 2| + 5t - 2}{4t}.$$

So, we have six possible options for an equilibrium position. In addition to these six, variants with the location of the leader firm at points 1 and 0 should be checked. The location of the follower firm is determined in this case from the response function of the follower firm. Equating the equation to zero and solving the resulting equations, we obtain:

$$x_2(x_1) = -\frac{\sqrt{(8t^2 - 16t)x_1 + 5t^2 - 4t + 4} - 4tx_1 - 3t + 2}{4t}$$

or

$$x_2(x_1) = \frac{\sqrt{(8t^2 - 16t)x_1 + 5t^2 - 4t + 4} + 4tx_1 + 3t - 2}{4t}.$$

For  $X_1 = 0$

$$x_2 = \frac{-\sqrt{5t^2 - 4t + 4} + 3t - 2}{4t}$$

or

$$x_2 = \frac{\sqrt{5t^2 - 4t + 4} + 3t - 2}{4t}$$

For  $X_1 = 1$ ;  $X_2 = 1$ .

Now let's get down to the most interesting thing, find which of the equilibrium values satisfy the conditions of the model. These values should maximize the integral functions of profit, and also satisfy the conditions  $0 \leq X_1 \leq X_2 \leq 1$ . Considering that the leader firm foresees the follower's actions at both stages, we will assume that the leader firm chooses from the possible equilibrium positions the one that maximizes its profit.

Table 1 below shows only the "permitted" locations of firms, in which the profit function of the leader firm is maximal, but the entire market is served. In this table, we took the transport tariff from 0.1 to 0.6.

**Approximate parameters of equilibrium placement:**

t	X1*	X2*	Π1*	Π2*	Stackelberg equilibrium
0,1	0,5	0,5	0,119	0,059	(2,40);(2,43)
0,2	0,5	0,5	0,113	0,056	(2,40);(2,43)
0,3	0,5	0,5	0,107	0,054	(2,40);(2,43)
0,4	0,451	0,541	0,102	0,051	(2,39);(2,42)
0,5	0,375	0,625	0,098	0,052	(2,39);(2,42)
0.6	0.338	0.697	0.099	0.055	(2.39);(2.42)

**Conclusion:**

The Nash – Stackelberg equilibrium of firms depends on transport costs. At low values of transport costs, firms are undifferentiated; at higher transport costs, firms are differentiated. Consequently, in Hotelling's model, the level of differentiation of firms depends on the transport tariff.

**References:**

- 1.Menshikov I.S. Lectures on game theory and economic modeling. M.: OOO "Contact Plus", 2010.
2. Bhadury J., Chandrasekaran R. Padmanabhan V. Competitive Location and Entry Deterrence in Hotelling's Duopoly Model // Location Science. 1994. V. 2. N. 4. P. 259-275.

3. Combes P.-Ph., Mayer T. and Thisse J.-F. Economic Geography: The Integration of Regions and Nations. Princeton: Princeton University Press, 2008.

4. Meza S., Tombak M. Endogenous Location Leadership // International Journal of Industrial Organization. 2009. V.27. P.687–707.

5. Prescott E.C., Visscher M. Sequential Location Among Firms with Foresight // Bell Journal of Economics. 1977.V.8.N2.P.378–393.

6. Stackelberg H. Marktform und Gleichgewicht. Wien und Berlin: J. Springer, 1934.

