Communication of subadditive measures on Jordan Banach algebra

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ABSTRACT
This article discusses the relationship between a subadditive measure and a trace on JBW–algebra.

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1. INTRODUCTION
It is known that continuation of a measure on idempotents of a JBW-algebra on the whole algebra is a trace and any state is connected with the trace using some positive element with the unit norm [1]. A similar fact for subadditive measures does not hold both on the Von Neumann algebra and in Jordan Banach algebras.

The article is devoted to finding good connections between a subadditive measure and a trace on JBW-algebras.

Empty A - JBW-algebra, □ - the logic of idempotents A.

Definition 1. A mapping \( m: \square \rightarrow [0, \infty] \) is called a subadditive measure on \( \square \) if it satisfies the following conditions:
1) \( m(\theta) = 0, \quad m(p) = 0 \Rightarrow p = \theta \)
2) \( p \leq q \Rightarrow m(p) \leq m(q) \)
3) \( p \sim q \Rightarrow m(p) = m(q) \)
4) \( m(p \lor q) \leq m(p) + m(q) \)
5) \( p_n \uparrow p \Rightarrow m(p_n) \rightarrow m(q) \)

Definition 2. A subadditive measure \( m \) is called finite if \( m(1) < +\infty \);

Semifinite if for any idempotent \( q \in \square, q \neq 0, q \leq p \) there exists an idempotent \( m(q) < +\infty \) such that.

Examples 1. The narrowing of the trace from A to \( \square \) is a subadditive measure.

2. Let \( \tau \) be a trace on A, \( \gamma \) a continuous map of \( \mathbb{R} \) to \( \mathbb{R} \) with the properties: \( \gamma(0) = 0, \quad \gamma(x) \leq \gamma(y) \) for \( x \leq y \) and \( \gamma(x + y) \leq \gamma(x) + \gamma(y) \) for all \( x, y \in \mathbb{R} \). Then \( m(p) = \gamma(\tau(p)) \) is a subadditive measure on.

Types of JBW - algebras are not related to the finiteness or semic finiteness of subadditive measures. For example, if a JBW - algebra is of type \( I_\infty \) with a semi-finite trace \( \tau \), then setting:

\[
m(p) = \frac{\tau(p)}{1 + \tau(p)}, \quad \forall p \in \square,
\]

We obtain a finite subadditive measure on \( \square \).
Definition 3. A mapping $l : A \to \mathbb{R}$ is called a normal semi-additive functional if

1) $l$ - positively;
2) $l(a) \leq l(b)$ at $a \leq b$, for all $a, b \in A$;
3) $l(a + b) \leq l(a) + l(b)$. For all $a, b \in A$;
4) $a_n \uparrow a \Rightarrow l(a_n) \to l(a)$.

Examples 1. A trace is a normal semi-additive functional.

2. For any $x \in A$ we put $l(x) = \| x \|$. It is easy to see that $l(x)$ is a semi-additive functional. The restriction to $\nabla$ has the form

$$m(p) = \begin{cases} 1, & \text{if } p \neq 0 \\
0, & \text{if } p = 0 \end{cases}$$

This is a subadditive measure on $\nabla$.

Theorem 4. Let $l$ be a normal semi-additive functional on $A$. If $l(U_s a) = l(a), a \in A$ for all, and $s$ is symmetry in $A$, then the restriction of the semi-additive functional $l$ from $A$ to $A$ is $\nabla$ a subadditive measure.

Evidence. Obviously, $l(p) = l(q)$ for all $p, q \in \nabla$ at $p \sim q$. Let then $p \leq q$. Then

$q = (q - p)\nabla p$

$$l(q) = l((q - p)\nabla p) = l((q - p) + p) \leq l(q - p) + l(p)$$

Means,

$$l(q) - l(p) \leq l(q - p)$$

Further, since for JBW-algebras it is known that $p\nabla q - p \sim q - p \wedge q$, then $l(p\nabla q - p) = l(q - p \wedge q)$ then. Therefore

$$l(p\nabla q) - l(p) \leq l(p\nabla q - p) = l(q - p \wedge q) \leq (q)$$

Consequently,

$$l(p\nabla q) \leq l(p) + l(q)$$

The theorem is proved.

The question arises whether the converse is true, that is, is it possible to extend any subadditive measure to $\nabla$ a semiadditive unitarily invariant functional on the whole algebra?

The following result was obtained in [2].

Let $A$ - JBW-algebra, $\nabla$ - many idempotents in $A$.

Definition 5. [2] A function $\mu : \nabla \to [0, \infty]$ is called a positive measure if

$$\mu(e + f) = \mu(e) + \mu(f)$$

For anyone $e, f \in \nabla$ with the condition $ef = 0$.

A measure is called a probability measure if $\mu(1) = 1$.

A positive measure on $\nabla$ is called a semiadditive measure if

$$\mu(e\nabla f) \leq \mu(e) + \mu(f), \text{ for all } e, f \in \nabla.$$
In this work, the authors require that the additivity condition on orthogonal idempotents is satisfied for subadditive measures; we consider this problem in the general case.

If a trace is given in the JBW - algebra A, then by connecting the subadditive measure with the trace, the continuation problem can be solved.

Theorem 7. Let A - JBW be an algebra of type II₁, ℰ an exact, normal, finite trace and m be a subadditive measure on A. Then there exists a continuous function ρ on R⁺ that is γ such that

\[ m(p) = ρ(ℰ(p)) \text{ for anyone } p ∈ ∆. \]

Theorem 8. If a JBW - algebra A is of type Iᵣ, then any subadditive measure m can be expressed in terms of ℰ using some semiadditive function.

Note: From the construction of the semiadditive function γ in the previous arguments, it is clear that it is not unique. When constructing the values of γ between points Sᵢ and Sᵢ₊₁, it suffices to take into account the fact that it satisfies the semi-additivity condition. In the simplest case, for example, for γ it is enough to replace the constant with a function between adjacent points [3].

From this reasoning it follows that every subadditive measure on ∆ the JBW - algebra A with a trace extends to a semiadditive unitarily invariant functional.

REFERENCES